Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 3, pp. 403–412, 1994

0021–8928/94 $24.00+0.00

0021-8928(94)00065-4

STOCHASTIC MECHANICAL SYSTEMS WITH A NON-RETAINING CONSTRAINT†

T. D. KONASHENKOVA and N. K. MOSHCHUK

Moscow

(*Received* 18 *September* 1992)

Holonomic mechanical systems with a non-retaining constraint and absolutely elastic collisions with the constraints subject to normal "white noise" perturbation forces are considered. The motion is defined and the problem of analysing such systems is formulated. It is shown that the Fokker–Planck–Kolmogorov equation for the one-dimensional probability distribution density of the state vector of systems free from a non-retaining constraint also governs the one-dimensional distribution density for the systems under consideration. Steady regimes in the restricted sense are studied and an approximate method of analysing such systems is proposed. Examples are considered.

When analysing stochastic vibratory impact systems most attention has been devoted to mechanical systems with one degree of freedom [1], which have been studied by the averaging method combined with the method of non-smooth transformations. The motion of some vibratory impact systems subject to random perturbations other than a white noise was studied in [2] using limit theorems.

1. Consider a holonomic mechanical system with an $(n+1)$-dimensional configuration space U ($\mathbf{q}=[q_0,\ q_1^{\mathrm{T}}]^{\mathrm{T}}$ denote the local coordinates in U and $\dim \mathbf{q}_1=n$), kinetic energy T, which is a quadratic form of the generalized velocities, potential energy $\Pi=\Pi(\mathbf{q})$, generalized force vector $\mathbf{F}$, and a non-retaining constraint $q_0\geqslant 0$. The Hamilton function of the system is equal to $H=T+\Pi$, where $T=\mathbf{p}^{\mathrm{T}}\Omega(\mathbf{q},\ t)\mathbf{p}/2$, and $\mathbf{p}$ is the generalized momentum vector. We shall assume that the generalized coordinates in U are chosen in such a way that

$$T=[\omega_0^2(\mathbf{q},t)p_0^2+\mathbf{p}_1^{\mathrm{T}}\Omega_1(\mathbf{q},t)\mathbf{p}_1]/2$$

(semi-geodesic coordinates) [3, 4]. It is assumed that the vector of generalized forces $\mathbf{F}$ can be represented in the form

$$\mathbf{F}=-\mathbf{D}(\mathbf{q},t)\mathbf{p}+\mathbf{b}(\mathbf{q},\mathbf{p},t)\mathbf{V}(t)$$

where $\mathbf{V}(t)$ is the vector representing normally distributed white noise with intensity matrix $\boldsymbol{\nu}(t)$, $\mathbf{D}$ and $\mathbf{b}$ are deterministic matrix-valued functions of the appropriate dimensions ($\mathbf{D}$ is a non-negative matrix). If the system was free from a non-retaining constraint, one could describe it by the Itô stochastic differential equations (SDEs)

$$\dot{\mathbf{q}}=\partial H/\partial\mathbf{p},\quad \dot{\mathbf{p}}=-\partial H/\partial\mathbf{q}-\mathbf{D}\mathbf{p}+\mathbf{b}\mathbf{V} \tag{1.1}$$

with appropriate initial conditions.

†*Prikl. Mat. Mekh.* Vol. 58, No. 3, pp. 32–40, 1994.

The main purpose of the present paper is to set up and analyse a stochastic mechanical system with a non-retaining constraint and absolutely elastic collisions with the constraint.

We define [3] an auxiliary system with $(n+1)$ degrees of freedom and free from a non-retaining constraint. To this end we change the variables in (1.1) as follows:

$$q_0(t)=|Q_0(t)|,\quad \mathbf{q}_1(t)=\mathbf{Q}_1(t),\quad p_0(t)=P_0\operatorname{sgn}Q_0(t),\quad \mathbf{p}_1(t)=\mathbf{P}_1(t) \tag{1.2}$$

These formulae define the mapping $(\mathbf{Q},\ \mathbf{P})\to(\mathbf{q},\ \mathbf{p})$, where $q_0\geqslant 0$.

The auxiliary system can be described by the following SDEs with the corresponding initial conditions

$$\begin{aligned}
&\dot{Q_0}=a_0,\quad \dot{\mathbf{Q}_1}=\mathbf{a}_1,\quad \dot{P_0}=a_2+\mathbf{b}_0\operatorname{sgn}Q_0\mathbf{V},\quad \dot{\mathbf{P}_1}=\mathbf{a}_3+\mathbf{b}_1\mathbf{V}\\
&a_0=\partial H^*/\partial P_0=\omega_0(|Q_0|,\mathbf{Q}_1,t)P_0,\quad \mathbf{a}_1=\partial H^*/\partial \mathbf{P}_1=\Omega_1(|Q_0|,\mathbf{Q}_1,t)\mathbf{P}_1\\
&a_2=-\partial H^*/\partial Q_0-\mathbf{D}_0(|Q_0|,\mathbf{Q}_1,t)[P_0,\mathbf{P}_1^{\mathrm{T}}\operatorname{sgn}Q_0]^{\mathrm{T}}\\
&\mathbf{a}_3=-\partial H^*/\partial \mathbf{Q}_1-\mathbf{D}_1(|Q_0|,\mathbf{Q}_1,t)[P_0\operatorname{sgn}Q_0,\mathbf{P}_1^{\mathrm{T}}]^{\mathrm{T}}\\
&\mathbf{b}_j=\mathbf{b}_j(|Q_0|,\mathbf{Q}_1,P_0\operatorname{sgn}Q_0,\mathbf{P}_1,t)\quad (j=0,1)
\end{aligned} \tag{1.3}$$

Here $\mathbf{D}_0$ and $\mathbf{b}_0$ are the first rows of $\mathbf{D}$ and $\mathbf{b}$, respectively, i.e. $\mathbf{D}=[\mathbf{D}_0,\mathbf{D}_1^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{b}=[\mathbf{b}_0,\ \mathbf{b}_1^{\mathrm{T}}]^{\mathrm{T}}$, and the Hamilton function H^* of the auxiliary system is defined as

$$H^*(\mathbf{Q},\mathbf{P})=H(|Q_0|,\mathbf{Q}_1,P_0,\mathbf{P}_1),\quad \partial H^*/\partial Q_0|_{Q_0=0}=\min\{0,\partial H/\partial Q_0|_{Q_0=+0}\}$$

In general, the drift and diffusion coefficients of the SDEs (1.3) undergo discontinuities of the first kind as the hyperplane $Q_0=0$ is crossed. In a similar case the problem of constructing the solution of the corresponding Fokker–Planck–Kolmogorov (FPK) equation for the one-dimensional probability density of the state vector was considered in [5].

The solution of the SDE (1.3) with the corresponding initial conditions is a Markov process $\mathbf{X}(t)\equiv[\mathbf{Q}(t)^{\mathrm{T}}\mathbf{P}(t)^{\mathrm{T}}]^{\mathrm{T}}$. We shall determine the motion of a stochastic mechanical system with a unilateral constraint and absolutely elastic collisions with the constraint as the stochastic process $\mathbf{x}(t)=[\mathbf{q}(t)^{\mathrm{T}}\mathbf{p}(t)^{\mathrm{T}}]^{\mathrm{T}}$ obtained from $\mathbf{X}(t)$ by (1.2). If the finite-dimensional distributions of $\mathbf{X}$ are known, then one can find all finite-dimensional distributions of the desired vector $\mathbf{x}$ using (1.2), i.e. solve the problem of analysis.

We note that the approach connected with determining a diffusion process with boundary screens using a certain auxiliary process (auxiliary system) is well known (see, for example, [6, p. 270]). However, for the problems of vibratory impact system mechanics a special approach is necessary because in this case an impact means an instantaneous jump from one point in phase space to another.

2. We shall study the question of what equation is satisfied by the one-dimensional density $f_1(\mathbf{x},\ t)$ of the process $\mathbf{x}(t)$ to be determined. To this end we introduce an involution of the phase space of the auxiliary (and the original) system with the aid of the formula $S\mathbf{X}=[-Q_0,\ \mathbf{Q}_1^{\mathrm{T}},\ -P_0,\ \mathbf{P}_1^{\mathrm{T}}]^{\mathrm{T}}(S^2=\mathrm{id})$. From the well-known formulae for the density of a random variable [7] one can find that the one-dimensional density $f_1(\mathbf{x},\ t)$ of the stochastic process $\mathbf{x}(t)$ with reflections is connected with the corresponding density $f_1^*(\mathbf{X},\ t)$ of the auxiliary process $\mathbf{X}(t)$ by

$$f_1(\mathbf{x},t)=f_1^*(\mathbf{x},t)+f_1^*(S\mathbf{x},t)\quad (q_0\geqslant 0) \tag{2.1}$$

It follows that

$$\partial f_1 / \partial q_0|_{q_0=0} = 0, \quad \partial f_1 / \partial p_0|_{p_0=0} = 0$$

The one-dimensional density $f_1^*(\mathbf{X}, t)$ of the auxiliary process satisfies the FPK equation (see, for example, [8])

$$\partial f_1^* / \partial t + G f_1^* = 0 \tag{2.2}$$

$$G(\cdot) = \frac{\partial^{\mathrm{T}}}{\partial \mathbf{X}}[\mathbf{a}(\mathbf{X})(\cdot)] - \frac{1}{2}\mathrm{tr}\left[\frac{\partial}{\partial \mathbf{P}}\frac{\partial^{\mathrm{T}}}{\partial \mathbf{P}}\sigma'(\mathbf{X})(\cdot)\right]$$

$$\mathbf{a} = [a_0, \mathbf{a}_1^{\mathrm{T}}, a_2, \mathbf{a}_3^{\mathrm{T}}]^{\mathrm{T}}, \quad \sigma' = \mathbf{b}'\nu\mathbf{b}'^{\mathrm{T}}, \quad \mathbf{b}' = [\mathbf{b}_0^{\mathrm{T}} \operatorname{sgn} Q_0, \mathbf{b}_1^{\mathrm{T}}]^{\mathrm{T}}$$

We will denote by G_S the operator obtained from G by replacing $\mathbf{X}$ by $S\mathbf{X}$.

Lemma 1. $G_S = G$

Proof. We have

$$G_S(\cdot) = -\frac{\partial^{\mathrm{T}}}{\partial Q_0}[a_0(S\mathbf{X})(\cdot)] + \frac{\partial^{\mathrm{T}}}{\partial \mathbf{Q}_1}[\mathbf{a}_1(S\mathbf{X})(\cdot)] - \frac{\partial^{\mathrm{T}}}{\partial P_0}[a_2(S\mathbf{X})(\cdot)] + \frac{\partial^{\mathrm{T}}}{\partial \mathbf{P}_1}[\mathbf{a}_3(S\mathbf{X})(\cdot)] -$$
$$-\frac{1}{2}\left\{\frac{\partial^2}{\partial P_0^2}[\nu\mathbf{b}_0^2(S\mathbf{X})(\cdot)] - \frac{\partial}{\partial P_0}\frac{\partial^{\mathrm{T}}}{\partial \mathbf{P}_1}[-\mathbf{b}_1(S\mathbf{X})\nu\mathbf{b}_0(S\mathbf{X})(\cdot)] - \right.$$
$$\left. -\mathrm{tr}\left[\frac{\partial}{\partial \mathbf{P}_1}\frac{\partial}{\partial P_0}(-\mathbf{b}_0(S\mathbf{X})\nu\mathbf{b}_1^{\mathrm{T}}(S\mathbf{X})(\cdot))\right] + \mathrm{tr}\left[\frac{\partial}{\partial \mathbf{P}_1}\frac{\partial^{\mathrm{T}}}{\partial \mathbf{P}_1}(\mathbf{b}_1(S\mathbf{X})\nu\mathbf{b}_1^{\mathrm{T}}(S\mathbf{X})(\cdot))\right]\right\}$$

Taking into account that

$$a_0(S\mathbf{X}) = -a_0(\mathbf{X}), \quad \mathbf{a}_1(S\mathbf{X}) = \mathbf{a}_1(\mathbf{X}), \quad a_2(S\mathbf{X}) = -a_2(\mathbf{X})$$
$$\mathbf{a}_3(S\mathbf{X}) = \mathbf{a}_3(\mathbf{X}), \quad \mathbf{b}_j(S\mathbf{X}) = \mathbf{b}_j(\mathbf{X}) \quad (j = 0, 1)$$

we obtain the required equality.

Assertion. The one-dimensional density $f_1(\mathbf{x}, t)$ of the process $\mathbf{x}(t)$ to be determined satisfies the FPK equation

$$\partial f_1 / \partial t + G' f_1 = 0, \quad G'(\cdot) = \frac{\partial^{\mathrm{T}}}{\partial \mathbf{q}}\left[\frac{\partial H}{\partial \mathbf{p}}(\cdot)\right] + \frac{\partial^{\mathrm{T}}}{\partial \mathbf{p}}\left[\left(-\frac{\partial H}{\partial \mathbf{q}} - \mathbf{D}\mathbf{p}\right)(\cdot)\right] - \frac{1}{2}\mathrm{tr}\left[\frac{\partial}{\partial \mathbf{p}}\frac{\partial^{\mathrm{T}}}{\partial \mathbf{p}}\sigma(\cdot)\right]$$
$$f_1(\mathbf{x}, t_0) = f_{10}(\mathbf{x}), \quad \sigma = \mathbf{b}\nu\mathbf{b}^{\mathrm{T}}$$

for the system without the confining non-retaining constraint with the normalization condition

$$\int_U f_1(\mathbf{x}, t)d\mathbf{x} = 1, \quad U = \{(\mathbf{q}, \mathbf{p}) | q_0 \geqslant 0\}$$

corresponding to a system with reflections.

Proof. Adding (2.2) to the equation

$$\partial f_1^*(S\mathbf{X}, t) / \partial t + G_S f_1^*(S\mathbf{X}, t) = 0$$

which can be obtained from (2.2) by substituting $S\mathbf{X}$ for $\mathbf{X}$, and using Lemma 1, we get

$$\partial / \partial t[f_1^*(\mathbf{X}, t) + f_1^*(S\mathbf{X}, t)] + G[f_1^*(\mathbf{X}, t) + f_1^*(S\mathbf{X}, t)] = 0$$

Hence

$$\partial f_1(\mathbf{X},t)/\partial t + Gf_1(\mathbf{X},t) = 0$$

It remains to observe that in the domain $q_0 \geq 0$ the operator G is identical with G' when $\mathbf{X}$ is replaced by $\mathbf{x}$. The assertion is proved. It generalizes the well-known similar result of [1, p. 352].

In a similar way one can prove Itô's formula for the process $\mathbf{x}(t)$ with reflections. Namely, let $\xi(\mathbf{x})$ be a twice differentiable function on U. then the stochastic differential of $\xi(\mathbf{x}(t))$ is

$$d\xi = \left[\frac{\partial^{\mathrm{T}}}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial^{\mathrm{T}}\xi}{\partial \mathbf{p}}\left(\frac{\partial H}{\partial \mathbf{q}} + \mathbf{Dp}\right) + \frac{1}{2}\operatorname{tr}\left(\sigma\frac{\partial^2\xi}{\partial \mathbf{p}^2}\right)\right]dt + \frac{\partial^{\mathrm{T}}\xi}{\partial \mathbf{p}}\mathbf{b}d\mathbf{W}$$

It has the same form as the stochastic differential of the process for the system with no non-retaining constraint.

Example 1. Consider the vertical motion (along the q-axis) of a material point in a uniform gravitational field inside a random medium with absolutely elastic reflections at $q=0$

$$\dot{q} = p, \quad \dot{p} = -\varepsilon p - k + V \quad (q \geq 0)$$

Here ε and k are dimensionless constants and V is a stationary normal white noise of intensity ν.

The assertion implies that the one-dimensional distribution density $f_1(q, p, t)$ satisfies the following FPK equation with the corresponding initial condition and the normalization condition

$$\partial f_1/\partial t + \partial(pf_1)/\partial q - \partial[(\varepsilon p + k)f_1]/\partial p - (\nu/2)\partial^2 f_1/\partial p^2 = 0 \tag{2.3}$$

$$f_1(q,p,t_0) = f_{10}(q,p), \quad \int_{-\infty}^{\infty}\int_0^{\infty} f_1(q,p,t)\,dq\,dp = 1$$

Note that when there is no reflecting screen at $q=0$ the equation for p can be separated and studied independently. However, in the corresponding auxiliary system

$$\dot{Q} = P, \quad \dot{P} = -\varepsilon P - k\,\mathrm{sgn}\,Q - \mathrm{sgn}\,QV$$

the equation for P can no longer be separated. In the example under consideration the FPK equation must therefore be set up for the whole state vector (q,p).

Example 2. Consider the motion of a simple pendulum between parallel walls (Fig. 1) in a uniform gravitational field. It is assumed that the pendulum is subjected to random and dissipative torques. We write down the equations of motion in terms of the dimensionless variables

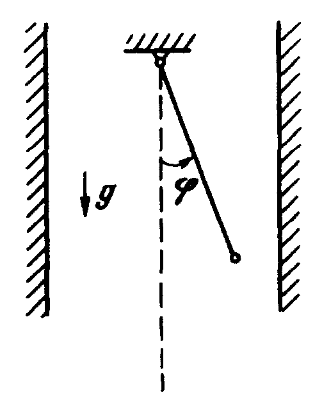

Fig. 1.

$$\varphi^{\cdot} = p, \quad p^{\cdot} = -\sin\varphi - \varepsilon p + b(\varphi)V(t)$$

Here $V(t)$ is stationary white noise of intensity ν and $\varepsilon > 0$ is the specific coefficient of friction. Collisions with the walls occur at $\varphi = \pm l$.

We shall show that in this case (with two unilateral constraints) the probability distribution density $f_1(\varphi, p, t)$ of the process with reflections satisfies the FPK equation set up as if there were no non-retaining constraints.

We introduce the functions $\delta(\Phi)$ and $\zeta(\Phi)$ [9] by

$$\delta = \begin{cases} \Phi, & \Phi \in [0, l] \\ -\Phi + 2l, & \Phi \in [l, 3l] \\ \Phi - 4l, & \Phi \in [3l, 4l] \end{cases}, \quad \zeta = \begin{cases} 1, & \Phi \in [0, l] \cup [3l, 4l] \\ -1, & \Phi \in (l, 3l) \end{cases}$$

We define the auxiliary system on a cylinder with coordinates $\Phi \in [0, 4l]$ (Φ is measured modulo $4l$) and $P \in R$ without non-retaining constraints by means of the substitution $\varphi = \delta(\Phi)$, $p = P\zeta(\Phi)$. It can be described by the following system of SDEs

$$\Phi^{\cdot} = P, \quad P^{\cdot} = -\zeta(\Phi)\sin\delta(\Phi) - \varepsilon P + b(\delta(\Phi))\zeta(\Phi)V(t) \tag{2.4}$$

We define an involution $\{\Phi, P\}$ of the phase space by $S\mathbf{X} = [2l - \Phi, -P]^{\mathrm{T}}$, $\mathbf{X} = [\Phi, P]^{\mathrm{T}}$. Then (2.1) holds for $\mathbf{x} = [\varphi, p]^{\mathrm{T}}$. Since the sign of the drift vector in (2.4) changes under this involution while that of the diffusion coefficient $\nu b^2\zeta^2$ does not, it follows that $G_S = G$, as in Section 1, and $f_1(\varphi, p, t)$ satisfies the FPK equation

$$\partial f_1 / \partial t + \partial(p f_1) / \partial\varphi + \partial[(-\sin\varphi - \varepsilon p) f_1] / \partial p - (\nu / 2)\partial^2 f_1 / \partial p^2 = 0$$

$$\int_{-\infty}^{\infty} \int_{-l}^{l} f_1(\varphi, p, t) d\varphi dp = 1, \quad f_1(\varphi, p, t_0) = f_{10}(\varphi, p)$$

The physical meaning of the assertion can be elucidated using the constructive approach to mechanical systems with non-retaining constraints involving the introduction of an auxiliary elastic force field of high rigidity in place of the non-retaining constraints [10]. We write down the FPK equations for the motion of the system without non-retaining constraints in the presence of the auxiliary field as well as when there is no such field. In the domain of motion with reflections (between the "walls") the drift and diffusion coefficients are the same for the two cases. Therefore the solutions of the FPK equations are also equal to one another apart from a constant. As the rigidity of the auxiliary field tends to infinity the probability that the trajectory is behind one of the walls tends to zero and for a motion with reflections the desired one-dimensional density should be normalized in the domain of admissible motion of the system with reflections. Of course this is not a rigorous argument, but it explains the somewhat formal approach of Section 1.

3. The assertion implies a close relationship between the steady regimes (SRs) of the stochastic system without non-retaining constraints and the SRs of the original system. The existence of SR in the system without constraints implies immediately that an SR exists in the system with non-retaining constraints (this is clear because a reflecting wall enhances the recurrent properties of the process), the one-dimensional densities of the systems being equal apart from multiplicative constant. In general, the converse is not true because the one-dimensional density of an SR of a system with reflections may fail to satisfy the normalization condition over the phase space of the system without constraints. In Example 1 there is an SR (i.e. the exact solution of (2.3) with $\partial f_1 / \partial t = 0$) with density

$$f_*(q, p) = \frac{2\varepsilon\sqrt{\varepsilon k}}{\nu\sqrt{\pi\nu}} \exp\left[-\frac{2\varepsilon}{\nu}\left(\frac{p^2}{2} + kq\right)\right] \quad (q \geqslant 0, \ p \in R) \tag{3.1}$$

However, no SR exists in the system without a reflecting screen. In Example 2 ($b \equiv 1$), if both walls are removed, then an SR exists with density

$$f_{st}(\varphi, p) = c \exp\left[-\frac{2\varepsilon}{\nu}\left(\frac{p^2}{2} - \cos\varphi\right)\right]$$

The SR remains after imposing two unilateral constraints. In this case the expression for f_{st} remains unchanged, except for the constant c.

Taking into account the connection between the existence of an SR in the original system and the system without constraints, to find the SRs in a stochastic mechanical system with absolutely elastic reflection one can use the methods for finding SRs developed for stochastic systems without non-retaining constraints (see, for example, [11]).

As an example, consider the motion of a massive strictly convex solid in a random medium above an absolutely smooth horizontal plane x, y, with z representing the vertical direction (the collisions are perfectly elastic). The motion occurs in a uniform gravitational field with acceleration g. Furthermore, r denotes the position vector of the centre of mass relative to the system of coordinates at rest. We assume that the random medium acts upon the body with a deterministic dissipative force and torque (which are linear functions of the velocities) as well as a random force and random torque. When there is no non-retaining constraint the equation of motion

$$\mathbf{r}^{\cdot\cdot} = \mathbf{g} - \varepsilon\mathbf{r}^{\cdot} + \mathbf{V}'$$

of the centre of mass is independent of the equation of motion

$$\mathbf{I}\boldsymbol{\omega}^{\cdot} + \boldsymbol{\omega} \times \mathbf{I}\boldsymbol{\omega}^{\cdot} = -\mathbf{D}\boldsymbol{\omega} + \mathbf{V}, \quad \boldsymbol{\gamma}^{\cdot} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0$$

of the body relative to the centre of mass. Here ε is the specific coefficient of friction, $\mathbf{I}$ is the inertia tensor, $\mathbf{D}'$ is a positive definite dissipation matrix, and $\mathbf{V}'$ and $\mathbf{V}$ are the independent vectors of normally distributed "white noise" with constant non-degenerate intensity matrices $\boldsymbol{\nu}'$ and ν. It is assumed for simplicity that $\boldsymbol{\nu}' = \nu_0\mathbf{E}$ ($\mathbf{E}$ is the unit matrix and $\nu_0 = \text{const}$).

It follows from the results obtained in [12] and the above assertion that if $\mathbf{D}$ is represented in the form

$$\mathbf{D} = \boldsymbol{\nu}(\lambda_1\mathbf{E} + \lambda_2\mathbf{I})$$

where λ_1 and λ_2 are real numbers such that the matrix $\lambda_1\mathbf{I} + \lambda_2\mathbf{I}^2$ is positive definite, then there an SR exists, the one-dimensional density of which is given by

$$f_{st}(\boldsymbol{\omega}, \boldsymbol{\gamma}, \mathbf{r}^{\cdot}, z) = c\exp(-\boldsymbol{\omega}^T\mathbf{I}\boldsymbol{\nu}^{-1}\mathbf{D}\boldsymbol{\omega} - \varepsilon\mathbf{r}^{\cdot 2}/\nu_0 - 2\varepsilon g z/\nu_0)$$

where f_{st} is normalized in the domain $U = R^3 \times S^2 \times R^3 \times [0, \infty)$. Thus

$$c = (\varepsilon/2)^{5/2}\pi^{-4}|\mathbf{I}\boldsymbol{\nu}^{-1}\mathbf{D}|^{1/2} g$$

4. The method involving the orthogonal expansion of the one-dimensional distribution density in a suitable Hilbert space [13] can be used for the approximate analysis of stochastic mechanical systems with a unilateral constraint. However, in this case the equations for the coefficients (quasimomenta) of the expansion of the one-dimensional density will differ from those in the case without a unilateral constraint in the system. This is so because $[M\xi(\mathbf{x})]$ is no longer equal to $MG^*\xi$ (M denotes the mathematical expectation and G^* is the adjoint operator to G).

We will use Examples 1 and 2 to illustrate this approach.

First we will consider the problem of a pendulum between parallel walls. The region $U = [-l, l] \times R$ with coordinates φ and p is the phase space of the system. In the Hilbert space $L_2(u, \mu)$ of functions on U

with measure $d\mu=\mu_0(p)d\varphi dp$, where $\mu_0=(2l)^{-1}(2\pi\gamma)^{-1/2}\exp[-p^2/(2\gamma)]$ (in what follows $\gamma>0$ is a constant), we define an orthonormal basis as the system of all products $\{e_nH_m,\ e_n'H_m\}$ of the trigonometric functions

$$1,e_n,e_n'\ (n=1,2,\ldots),\ \ e_n=\sqrt{2}(-1)^n\sin\frac{(2n-1)\pi\varphi}{2l},\ \ e_n'=\sqrt{2}(-1)^n\cos\frac{n\pi\varphi}{l} \tag{4.1}$$

and Hermite polynomials

$$H_0(p)=1,\ \ H_1(p)=\frac{p}{\sqrt{\gamma}},\ldots,\ \ H_m(p)=(-1)^m\sqrt{\frac{\gamma^m}{m!}}\exp\left(\frac{p^2}{2\gamma}\right)\frac{d^m}{dp^m}\exp\left(-\frac{p^2}{2\gamma}\right)$$

The desired one-dimensional density $f_1(\varphi,p,t)$ can be represented as

$$f_1(\varphi,p,t)=\mu_0\sum_{m=0}^{\infty}H_m\left[d_m+\sum_{n=1}^{\infty}(a_{mn}e_n+b_{mn}e_n')\right] \tag{4.2}$$

at any instant of time, the coefficients (the quasimomenta of the desired process $[\varphi,\ p]^{\mathrm{T}}$) being

$$d_m=MH_m(p),\ \ a_{mn}=MH_m(p)e_n(\varphi),\ \ b_{mn}=MH_m(p)e_n'(\varphi) \tag{4.3}$$

Note that system (4.1) is chosen because, in general, f_1 has different values at the points $\varphi=l$ and $\varphi=-l$. The traditional trigonometric system is therefore unsuitable here.

Lemma 2. For any smooth function $\xi(\varphi,p)$

$$(M\xi)^{\cdot}=M(G^*\xi)-\int_{-\infty}^{\infty}p[f(l,p,t)\xi(l,p)-f(-l,p,t)\xi(-l,p)]dp \tag{4.4}$$

$$G^*=p\frac{\partial}{\partial\varphi}-(\sin\varphi+\varepsilon p)\frac{\partial}{\partial p}+\frac{\nu}{2}\frac{\partial^2}{\partial p^2}$$

provided that all the mathematical expectations and integrals in (4.4) exist.

Proof. By definition

$$M\xi(\varphi,p)=\int_{-\infty}^{\infty}\int_{-l}^{l}f_1(\varphi,p,t)\xi(\varphi,p)d\varphi dp$$

Using the FPK equation for f_1 and the formula for integration by parts, we have

$$(M\xi)^{\cdot}=\int_{-\infty}^{\infty}\int_{-l}^{l}\frac{\partial f_1}{\partial t}\xi d\varphi dp=\int_{-\infty}^{\infty}\int_{-l}^{l}\left\{-\frac{\partial(f_1p)}{\partial\varphi}+\frac{\partial[(\sin\varphi+\varepsilon p)f_1]}{\partial p}+\frac{\nu}{2}\frac{\partial^2 f_1}{\partial p^2}\right\}\xi d\varphi dp=$$

$$=\int_{-\infty}^{\infty}p[f_1(l,p,t)\xi(l,p)-f_1(-l,p,t)\xi(-l,p)]dp+\int_{-\infty}^{\infty}\int_{-l}^{l}f_1G^*\xi d\varphi dp$$

i.e. we obtain the required equality (4.4).

Using Lemma 4 and formulae (4.2) and (4.3), we obtain the following linear differential equations with constant coefficients for the coefficients d_m, a_{mn}, b_{mn} of (4.2) (summation is with respect to k from $k=1$ to ∞)

$$\dot{a}_{mn}=-\varepsilon ma_{mn}-\sqrt{\frac{2m}{\gamma}}\frac{\cos l}{l}\left\{1-\left[(2n-1)\frac{\pi}{2l}\right]^2\right\}^{-1}d_{m-1}-$$

$$-\frac{2\sqrt{m\gamma}}{l}\sum\left(\eta_{kn}+\frac{\cos l}{4\gamma}\kappa_{kn}\right)b_{m-1,k}-\frac{2}{l}\sqrt{(m+1)\gamma}\sum\eta_{kn}b_{m+1,k}$$

$$\dot{b}_{mn} = -\varepsilon m b_{mn} - \frac{2\sqrt{m\gamma}}{l}\sum a_{m-1,k}\left(\eta'_{kn} + \frac{\cos l}{4\gamma}\kappa'_{kn}\right) - \frac{2}{l}\sqrt{(m+1)\gamma}\sum \eta'_{kn}a_{m+1,k}$$

$$\dot{d}_m = -\varepsilon m d_m - \sum\sqrt{\frac{2m}{\gamma}}\frac{4l\cos l}{[4l^2 - \pi^2(2k-1)^2]}a_{m-1,k} + \frac{\sqrt{2(m+1)\gamma}}{l}\sum a_{m+1,k} + \frac{\sqrt{2m\gamma}}{l}\sum a_{m-1,k}$$

$$\eta_{kn} = \left(\frac{2k}{2n-1}\right)^2\left[1-\left(\frac{2k}{2n-1}\right)^2\right]^{-1}, \quad \eta'_{kn} = \left(\frac{2k-1}{2n}\right)^2\left[1-\left(\frac{2k-1}{2n}\right)^2\right]^{-1}$$

$$\kappa_{kn} = \alpha^-(k) + \alpha^-(-k) + \alpha^+(k) + \alpha^+(-k)$$

$$\kappa'_{kn} = \beta^+(-k) + \alpha^-(-k) + \alpha^+(k) + \beta^-(k)$$

$$\alpha^{\pm}(k) = \left[1 \pm (2n-1)\frac{\pi}{2l} + \frac{k\pi}{l}\right]^{-1}$$

$$\beta^{\pm}(k) = \left[1 \pm (2n+1)\frac{\pi}{2l} + \frac{k\pi}{l}\right]^{-1}, \quad \gamma = \frac{\nu}{2\varepsilon}$$

Considering only those m, n in (4.2) for which $0 \leqslant m,\ n \leqslant N$ (the orthogonal summation method for the series (4.2)) and setting the remaining c_{mn} equal to zero, we obtain a system of linear differential equations of order $N(2N+3)$. Solving this system (for example on a computer), we obtain an approximate expression for the distribution density and approximate values of the quasimomenta at any instant of time.

The results of numerical computer experiments for $N = 6$ are presented in Fig. 2 (in which the one-dimensional density of the SR is shown) and Fig. 3 (in which the time dependence of Mp, Dp, $M\sin\varphi$, and $M\cos\varphi$ is presented).

Now let us consider Example 1. The half-plane $U = [0, \infty) \times R$ with coordinates q and p is the phase space of the system. In the Hilbert space $L_2(U, \mu)$ of functions on U with measure μ such that

$$d\mu = \mu_1(q)\mu_1(p)dqdp, \quad \mu_1(q) = \exp(-q), \quad \mu_2 = (2\pi\gamma)^{-1/2}\exp[-p^2/(2\gamma)]$$

we define an orthonormal basis to be the system of all products of Laguerre polynomials

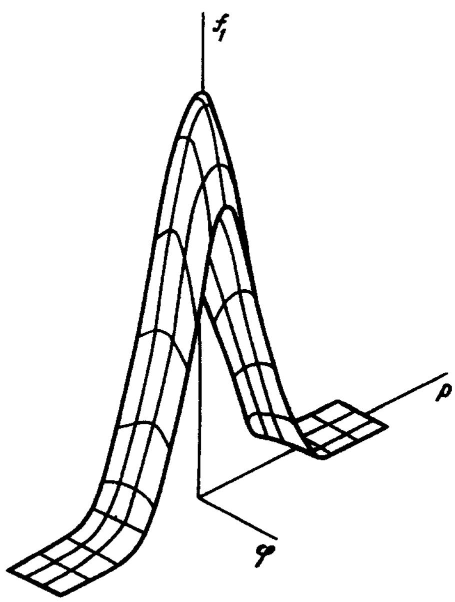

Fig. 2.

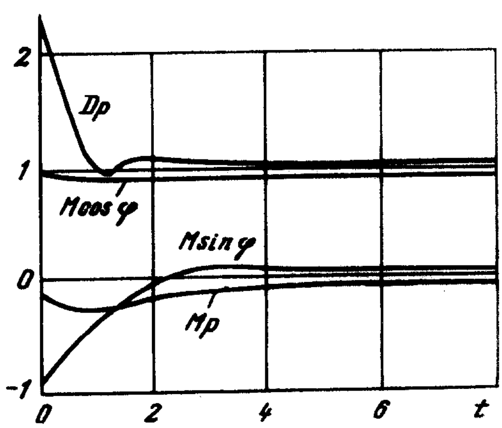

Fig. 3.

$$L_n(q)=(n!)^{-1}e^q\frac{d^n}{dq^n}(q^ne^{-q})=\sum_{j=0}^{n}(-1)^jc_n^jj^{-1}q^j,\quad C_n^j=\frac{n!}{j!(n-j)!}\quad (n=0,1,\ldots)$$

and Hermite polynomials $H_m(p)$.

The desired one-dimensional density $f_1(q,p,t)$ can be represented as

$$f_1(q,p,t)=\mu_1(q)\mu_2(p)\sum_{m,n=0}^{\infty}c_{mn}H_m(p)L_n(q) \tag{4.5}$$

It can be shown that Lemma 2 with $q=0$ and $q=\infty$ substituted, respectively, for $-l$ and l in (4.4) holds for any smooth function $\xi(q,p)$. Hence, taking (4.5) into account, we can obtain the following equations for the coefficients of the expansion c_{mm}

$$\dot{c}_{mn}=MG^*H_m(p)L_n(q)+\int_{-\infty}^{\infty}pH_m(p)L_n(0)f_1(0,p,t)dp=$$

$$=-\varepsilon mc_{mn}+\sqrt{(m+1)\gamma}\sum_{j=n}^{\infty}c_{m+1,j}+\sqrt{m\gamma}\sum_{j=n}^{\infty}c_{m-1,j}-k\sqrt{\frac{m}{\gamma}}c_{m-1,n} \tag{4.6}$$

Applying the triangular summation method to series (4.5), i.e. restricting oneself only to those c_{mn} for which $0\leqslant m+n\leqslant N$, and setting the others equal to zero, from (4.6) we obtain the precise expressions for the coefficients of the expansion of the stationary density (3.1) (the only non-zero coefficients are $c_{0n}=(1-\gamma/k)^n$). Using the same summation method and solving Eq. (4.6), we can obtain approximate expressions for the non-stationary probability distribution density and approximate values for the quasi-momenta at any instant of time.

Note that the application of the orthogonal expansion method to systems with unilateral constraints requires an additional justification because of the effect reflected in Lemma 2.

Note that the orthogonal expansion method can be applied both to the system without non-retaining constraints (followed by normalization in the domain of possible motions of the system with reflections) and to the original system with collisions. In these cases the orthonormal bases will be different (and so will the phase spaces). For example, in the problem of a simple pendulum moving in a random medium with two constraints one can use an orthonormal basis, which is a combination of Fourier trigonometric functions and Hermite polynomials and seek $f_1:S\times R\to R$ without regard to the constraints (as in [13]). Then one only needs to adjust d_0 using a different normalization condition. Computational experiments have also been carried out for this approach, confirming the results presented in Figs 2 and 3.

In conclusion, we note that the results can be easily carried over to the case when the vector of random forces in (1.1) can be represented in the form $\mathbf{b}(\mathbf{q},\mathbf{p},t)\boldsymbol{\pi}(t)$, and $\boldsymbol{\pi}(t)$ is a random vector that satisfies the forming filter equation [8]

$$\boldsymbol{\pi}^{\cdot}=\mathbf{a}(\boldsymbol{\pi})+\mathbf{b}(\boldsymbol{\pi})\mathbf{V}(t)$$

REFERENCES

1. DIMENTBERG M. F., *Non-linear Stochastic Problems of Mechanical Oscillations.* Nauka, Moscow, 1980
2. KOVALEVA A. S., Application of asymptotic methods to some stochastic problems in vibratory impact system dynamics. *Prikl. Mat. Mekh.* **48**, 5, 733–737, 1984.
3. IVANOV A. P. and MARKEYEV A. P., On the dynamics of systems with unilateral constraints. *Prikl. Mat. Mekh.* **48**, 4, 632–636, 1984
4. RASHEVSKII P. K., *A Course in Differential Geometry.* Gostekhizdat, Moscow, 1950.
5. KHAZEN E. M., *Optimum Stochastic Solution Methods and Optimal Control Problems.* Sovetskoye Radio, Moscow, 1968.
6. PROKHOROV Yu. V. and ROZANOV Yu. A., *The Theory of Probability. Fundamental Ideas. Limit Theorems. Stochastic Processes.* Nauka, Moscow, 1987.
7. PUGACHEV V. S., *Probability Theory and Mathematical Statistics.* Nauka, Moscow, 1979.
8. PUGACHEV V. S. and SINITSYN I. N., *Stochastic Differential Systems.* Nauka, Moscow, 1990.

9. ZHURAVLEV V. F. and KLIMOV D. M., *Applied Methods in Oscillation Theory*. Nauka, Moscow, 1988.
10. KOZLOV V. V., A constructive method of justifying the theory of systems with non-retaining constraints. *Prikl. Mat. Mekh.* **52**, 6, 883–894, 1988
11. MOSHCHUK N. K. and SINITSYN I. M., Steady regimes and regimes reducible to steady ones in normal stochastic differential systems. *Prikl. Mat. Mekh.* **55**, 6, 895–903, 1991
12. MOSHCHUK N. K. and SINITSYN I. N., The fluctuations of a body with a fixed point in a random medium. *Izv. Akad. Nauk SSSR, MTT* 1, 39–44, 1993
13. MOSHCHUK N. K., An approximate method of analysing stochastic mechanical systems. *Prikl. Mat. Mekh.* **56**, 6, 907–917, 1992.

Translated by T.J.Z.